Analytical Evaluation of the Nonlinear Vibration of Coupled Oscillator Systems

Mahmoud Bayat[a], Mehran Shahidi[a], Amin Barari[b], and Ganji Domairry[c]

[a] Department of Civil Engineering, Shomal University, P. O. Box 731, Amol, Iran
[b] Department of Civil Engineering, Aalborg University, Sohngårdsholmsvej 57, 9000 Aalborg, Aalborg, Denmark
[c] Department of Mechanical Engineering, Babol University of Technology, Babol, Iran

Reprint requests to A. B.; E-mail: ab@civil.aau.dk or amin78404@yaho.com

We consider periodic solutions for nonlinear free vibration of conservative, coupled mass-spring systems with linear and nonlinear stiffnesses. Two practical cases of these systems are explained and introduced. An analytical technique called energy balance method (EBM) was applied to calculate approximations to the achieved nonlinear differential oscillation equations where the displacement of the two-mass system can be obtained directly from the linear second-order differential equation using the first order of the current approach. Compared with exact solutions, just one iteration leads us to high accuracy which is valid for a wide range of vibration amplitudes as indicated in the presented examples.

Key words: Coupled Nonlinear Oscillators; Nonlinear Stiffness; Duffing Equation; Energy Balance Method.

1. Introduction

In general, vibrations are undesirable. Their effects on mechanical systems are injurious and can cause costly failures. Therefore, vibrations and their effects need to be suppressed. Some dynamic systems that require two independent coordinates, or degrees of freedom, to describe their motion, are called two degree of freedom systems (TDOF). There has been an increasing attention in recent years towards the motion of the nonlinear TDOF oscillation system [1 – 5]. Coupled vibrating systems such as elastic beams supported by two springs and vibration of a milling machine can be modeled using those TDOF systems [6]. Their equations of motion consist of two second-order differential equations with cubic nonlinearities.

Due to many usages of TDOF systems, some of which were mentioned above, solving the equations of motion for a mechanical system associated with linear and nonlinear properties was attempted through the transformation into a set of differential algebraic equations using intermediate variables; here the equations of motion for a TDOF system are transformed into the Duffing equation [7].

Many analytical and numerical approaches have been investigated due to the limitation of existing exact solutions. The Duffing equation is one of the useful nonlinear equations which can be solved using different kinds of analytical approximations, developed for solving these complicated nonlinear systems.

These techniques include the harmonic balance [8], homotopy perturbation [9 – 12], modified Lindstedt-Poincaré [13], Adomian decomposition [14], parameter-expansion [15, 16], parameterized perturbation [17], multiple scale [18], energy balance [19, 20], variational approach [21, 22], variational iteration [23 – 26], Newton-harmonic balancing [27], differential transformation [28], and max-min [29 – 31] method.

He [32, 33] gave a comprehensive review of the recently developed nonlinear analytical techniques for solving nonlinear oscillation problems, which comprise the relatively newer family of solutions which lie within the framework of periodic analytical solutions. Other methods have also been developed in recent years which seem to be promising in obtaining accurate solutions to the generally more difficult nonlinear oscillation problems. He's energy balance method [34 – 39] is one such method, which is actually a heuristic approach valid not only for weakly nonlinear systems, but also for strongly nonlinear ones [34]. In this paper He's energy balance method is

employed to solve the governing nonlinear oscillator problems.

Two examples [3, 4] are considered in order to demonstrate the application of the method to actual oscillation problems, and to assess the capability of the methods in arriving at accurate solutions of the problem.

2. Procedures of Solving the Duffing Equation Using the Energy Balance Method

The conservative autonomous system of a cubic Duffing equation is represented by the following second-order differential equation:

$$\ddot{v}(t) + \alpha v(t) + \beta v(t)^3 = 0 \qquad (1)$$

with the initial conditions

$$v(0) = A, \quad \dot{v}(0) = 0, \qquad (2)$$

in which v and t are generalized dimensionless displacement and time variables, respectively, and α and β are any positive constant parameters.

In the present paper, we consider a general nonlinear oscillator in the form

$$\ddot{v}(t) + f(v(t)) = 0. \qquad (3)$$

Its variational principle can be easily established using the semi-inverse method:

$$J(v) = \int_0^t \left(-\frac{1}{2}\dot{v} + F(v) \right) dt. \qquad (4)$$

It is Hamiltonian, therefore, can be written in the form

$$H = \frac{1}{2}\dot{v} + F(v) = F(A) \qquad (5)$$

or

$$R(t) = H = \frac{1}{2}\dot{v} + F(v) = F(A). \qquad (6)$$

Oscillation systems contain two important physical parameters, i. e. the frequency ω and the amplitude of oscillation A. So let us consider the initial conditions

$$v(0) = A, \quad \dot{v}(0) = 0 \qquad (7)$$

and use the following trial function to determine the angular frequency ω:

$$v(t) = A\cos\omega t. \qquad (8)$$

Substituting (8) into v term of (6), yields

$$R(t) = \frac{1}{2}\omega^2 A^2 \sin^2\omega t + F(A\cos\omega t) - F(A) = 0. \qquad (9)$$

If by chance, the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by the appropriate choice of ω. Since (8) is only an approximation the exact solution R cannot be made zero everywhere. Collocation at $\omega t = \frac{\pi}{4}$ gives

$$\omega = \sqrt{\frac{2(F(A)) - F(A\cos\omega t)}{A^2 \sin^2\omega t}}. \qquad (10)$$

Its period can be written in the form

$$T = \frac{2\pi}{\sqrt{\frac{2(F(A)) - F(A\cos\omega t)}{A^2 \sin^2\omega t}}}. \qquad (11)$$

3. Examples of Nonlinear two Degree of Freedom (TDOF) Oscillating Systems

In this section, two practical examples of TDOF oscillation systems are illustrated to show the applicability, accuracy, and effectiveness of the proposed approach. It is noted that the subscripts 1, 2, 3 and $\omega, u(t), v(t), x(t)$, and $y(t)$ denote the first-order analytical approximations in this section.

Example 1. Consider the two-mass system model as shown in Figure 1.

The equations of motion are given as [3]

$$m\ddot{x} + k_1(x - y) + k_2(x - y)^3 = 0, \qquad (12)$$

$$m\ddot{y} + k_1(y - x) + k_2(y - x)^3 = 0 \qquad (13)$$

with the initial conditions

$$x(0) = X_0, \quad x(0) = 0, \qquad (14)$$

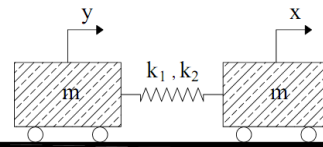$$y(0) = Y_0, \quad \dot{y}(0) = 0, \qquad (15)$$

Fig. 1. Two masses connected by linear and nonlinear stiffnesses.

where double dots in (12) and (13) denote double differentiation with respect to time. t, k_1, and k_2 are linear and nonlinear coefficients of the spring stiffness, respectively. Dividing (12) and (13) by mass m yields

$$\ddot{x} + \frac{k_1}{m}(x-y) + \frac{k_2}{m}(x-y)^3 = 0, \tag{16}$$

$$\ddot{y} + \frac{k_1}{m}(y-x) + \frac{k_2}{m}(y-x)^3 = 0. \tag{17}$$

Introducing the intermediate variables u and υ as follows [7]:

$$x := u, \tag{18}$$

$$y - x := \upsilon \tag{19}$$

and transforming (16) and (17) yields

$$\ddot{u} - \kappa \upsilon - \rho \upsilon^3 = 0, \tag{20}$$

$$\ddot{\upsilon} + \ddot{u} - \kappa \upsilon + \rho \upsilon^3 = 0, \tag{21}$$

where $\kappa = k_1/m$ and $\rho = k_2/m$. Equation (19) is rearranged as follows:

$$\ddot{u} = \kappa \upsilon + \rho \upsilon^3. \tag{22}$$

Substituting (22) into (21) yields

$$\ddot{\upsilon} + 2\kappa \upsilon + 2\rho \upsilon^3 = 0 \tag{23}$$

with the initial conditions

$$\upsilon(0) = y(0) - x(0) = Y_0 - X_0 = A, \quad \dot{\upsilon}(0) = 0. \tag{24}$$

Thus, (23) is equivalent to the Duffing equation (1) with $\alpha = 2\kappa$ and $\beta = 2\rho$. For solving (23) using an energy balance method, the approximate solutions of $\upsilon(t)$ can be substituted back into (22) to obtain the intermediate variable $u(t)$ by double integration.

3.1. First-Order Analytical Approximation for (23)

For (23) the variational formulation can be readily obtained as follows:

$$J(\upsilon) = \int_0^t \left(-\frac{1}{2}\dot{\upsilon}^2 + \kappa \upsilon^2 + \frac{1}{2}\rho \upsilon^4 \right) dt. \tag{25}$$

Its Hamiltonian, therefore, can be written in the form

$$H = -\frac{1}{2}\dot{\upsilon}^2 + \kappa \upsilon^2 + \frac{1}{2}\rho \upsilon^4 \tag{26}$$

and

$$H_{t=0} = \kappa A^2 + \frac{1}{2}\rho A^4, \tag{27}$$

$$H_t - H_{t=0} = \frac{1}{2}\dot{\upsilon}^2 + \kappa \upsilon^2 + \frac{1}{2}\rho \upsilon^4 - \kappa A^2 - \frac{1}{2}\rho A^4. \tag{28}$$

We will use the trial function to determine the angular frequency ω, i. e.

$$\upsilon(t) = A \cos \omega t. \tag{29}$$

If we substitute (29) into (28), it results the following residual equation:

$$\frac{1}{2}(-A\omega \sin(\omega t))^2 + \kappa(A\cos(\omega t))^2 + \frac{1}{2}\rho(A\cos(\omega t))^4 - \kappa A^2 - \frac{1}{2}\rho A^4 = 0. \tag{30}$$

If we collocate at $\omega t = \frac{\pi}{4}$, we obtain

$$\frac{1}{4}A^2\omega^2 - \frac{1}{2}\kappa A^2 - \frac{3}{8}\rho A^4 = 0. \tag{31}$$

This leads to the following result:

$$\omega = \frac{1}{2}\sqrt{8\kappa + 6\rho A^2}. \tag{32}$$

According to (8) and (32), we can obtain the following approximate solution:

$$\upsilon(t) = A \cos\left(\frac{1}{2}t\sqrt{8\kappa + 6\rho A^2} \right). \tag{33}$$

The first-order analytical approximation for $u(t)$ is

$$u(t) = \iint \left(\kappa \upsilon + \rho \upsilon^3 \right) dt dt = \frac{1}{9\omega^2}A\cos(\omega t)(9\kappa + \rho A\cos^2(\omega t) + 6\rho A^2). \tag{34}$$

Therefore, the first-order analytical approximate displacements $x(t)$ and $y(t)$ are

$$x(t) = u(t), \tag{35}$$

$$y(t) = u(t) + A\cos(\omega t). \tag{36}$$

Example 2. Consider a two-mass system connected with linear and nonlinear springs and fixed to a body at two ends as shown in Figure 2.

The equation of motion is given as [4]

$$m\ddot{x} + k_1 x + k_1(x-y) + k_2(x-y)^3 = 0, \tag{37}$$

$$m\ddot{y} + k_1 y + k_1(y-x) + k_2(y-x)^3 = 0 \tag{38}$$

with the initial conditions

$$x(0) = X_0, \quad x(0) = 0, \tag{39}$$
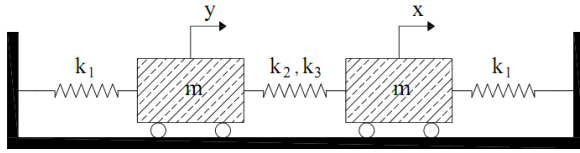
$$y(0) = Y_0, \quad \dot{y}(0) = 0, \tag{40}$$

Fig. 2. Two-mass system connected with the fixed body.

where double dots in (37) and (38) denote double differentiation with respect to time t, k_1 and k_2 are linear coefficients of the spring stiffness, and k_3 is the nonlinear coefficient of the spring stiffness. Dividing (37) and (38) by mass m yields

$$\ddot{x} + \frac{k_1}{m}x + \frac{k_2}{m}(x-y) + \frac{k_3}{m}(x-y)^3 = 0, \qquad (41)$$

$$\ddot{y} + \frac{k_1}{m}y + \frac{k_2}{m}(y-x) + \frac{k_3}{m}(y-x)^3 = 0. \qquad (42)$$

Like in Example 1, transforming the above equations using intermediate variables in (18) and (19) yields

$$\ddot{u} + \gamma u - \eta v - \lambda v^3 = 0, \qquad (43)$$

$$\ddot{u} + \ddot{v} + \gamma u + \gamma v + \eta u + \lambda v^3 = 0, \qquad (44)$$

where $\gamma = k_1/m$, $\eta = k_2/m$, and $\lambda = k_3/m$. Rearranging (43) as

$$\ddot{u} = \eta v + \lambda v^3 - \gamma u \qquad (45)$$

and substituting back into (44) yields

$$v + 2\lambda v^3 = 0 \qquad (46)$$

with the initial conditions

$$v(0) = y(0) - x(0) = Y_0 - X_0 = A, \quad \dot{v}(0) = 0. \quad (47)$$

Equation (46) is again equivalent to the Duffing equation (1) with $\alpha = \gamma + 2\eta$ and $\beta = 2\lambda$. For solving (46) using the energy balance method, the approximate solutions of $v(t)$ can be substituted back into (45) to yield

$$\ddot{v} + \gamma u = \eta v + \lambda v^3 \qquad (48)$$

with the initial conditions

$$u(0) = x(0) = X_0, \quad \dot{u}(0) = 0. \qquad (49)$$

Equation (48) is a linear non-homogeneous second-order ordinary differential equation and it can be solved readily using a standard method such as the Laplace transformation.

3.2. First-Order Analytical Approximation for (46)

The variational formulation can be readily obtained for (46) as follows:

$$J(v) = \int_0^t \left(-\frac{1}{2}\dot{v}^2 + \frac{1}{2}(\gamma + 2\eta)v^2 + \frac{1}{2}\lambda v^4 \right) dt. \quad (50)$$

It's Hamiltonian, therefore, can be written in the form

$$H = \frac{1}{2}\dot{v}^2 + \frac{1}{2}(\gamma + 2\eta)v^2 + \frac{1}{2}\lambda v^4 \qquad (51)$$

and

$$H_{t=0} = \frac{1}{2}\lambda A^4 + \frac{1}{2}(\gamma + 2\eta)A^2, \qquad (52)$$

$$\begin{aligned} H_t - H_{t=0} = {} & \frac{1}{2}\dot{v}^2 + \frac{1}{2}(\gamma + 2\eta)v^2 + \frac{1}{2}\lambda v^4 \\ & - \frac{1}{2}\lambda A^4 + \frac{1}{2}(\gamma + 2\eta)A^2. \end{aligned} \qquad (53)$$

We will use the trial function to determine the angular frequency ω, i. e.

$$v(t) = A\cos\omega t. \qquad (54)$$

If we substitute (54) into (53), it results in the following residual equation:

$$\begin{aligned} & \frac{1}{2}(-A\omega\sin(\omega t))^2 + \frac{1}{2}(\gamma + 2\eta)(A\cos(\omega t))^2 \\ & + \frac{1}{2}\lambda(A\cos(\omega t))^4 - \frac{1}{2}\lambda A^4 + \frac{1}{2}(\gamma + 2\eta)A^2 = 0. \end{aligned} \qquad (55)$$

If we collocate at $\omega t = \frac{\pi}{4}$, we obtain

$$\frac{1}{4}A^2\omega^2 - \frac{1}{4}(\gamma + 2\eta)A^2 - \frac{3}{8}\lambda A^4 = 0, \qquad (56)$$

This leads to the following result:

$$\omega = \frac{1}{2}\sqrt{6\lambda A^2 + 8\eta + 4\gamma}. \qquad (57)$$

According to (8) and (57), we can obtain the following approximate solution:

$$v(t) = A\cos\left(\frac{1}{2}t\sqrt{6\lambda A^2 + 8\eta + 4\gamma}\right). \qquad (58)$$

By (48), the first-order analytical approximation for $u(t)$ is

$$\begin{aligned} u(t) = {} & \cos(\sqrt{\gamma}t)\left[\lambda A^3\gamma - 7\lambda A^3\omega^2 - \eta\gamma A + 9\eta\omega^2 A \right. \\ & \left. + 5\gamma^2 - 50\gamma\omega^2 + 45\omega^4\right]\left[\gamma^2 - 10\gamma\omega^2 + 9\omega^4\right]^{-1} \\ & - 36\left[-\frac{1}{36}(\omega^2 - \gamma)\lambda A^2\cos(3\omega t) + \cos(\omega t)\left(\eta - \frac{3}{4}\lambda A^2\right)\right. \\ & \left. \cdot(\omega^2 - \frac{1}{9}\gamma)\right]A\left[4\gamma^2 - 40\gamma\omega^2 + 36\omega^4\right]^{-1}. \quad (59) \end{aligned}$$

Table 1. Comparison of angular frequencies in (23) with the exact solution.

Constants					Results		
m	k_1	k_2	X_0	Y_0	Energy balance ω	Exact solution ω_e	Relative error
1	5	5	5	1	11.4018	11.1921	0.2097
1	1	1	10	-5	18.4255	18.0302	0.3953
5	10	10	20	30	17.4356	17.0672	0.3684
10	50	-0.01	-20	40	2.1448	2.0795	0.0653
1	10	5	20	25	14.4049	14.1514	0.2535
100	200	300	400	200	424.2688	415.053	9.2158

Table 2. Comparison of angular frequencies in (46) with the exact solution.

Constants					Results		
m	k_1	k_2	X_0	Y_0	Energy balance ω	Exact solution ω_e	Relative error
1	1	1	1	5	5.1961	5.1078	0.0883
1	1	1	5	5	13.8022	13.5121	0.2901
5	10	20	30	-10	60.0833	58.7856	1.2977
10	50	70	90	20	220.4972	215.7113	4.7859
1	25	20	-0.05	-10	1.8708	1.8413	0.0295
100	200	300	400	-50	244.9653	239.6455	5.3198

Therefore, the first-order analytical approximate displacements $x(t)$ and $y(t)$ are

$$x(t) = u(t), \tag{60}$$

$$y(t) = u(t) + A\cos(\omega t). \tag{61}$$

4. Discussion of Examples

To illustrate and verify the accuracy of the energy balance method, the comparison with published data and exact solutions is presented. The exact frequency ω_e for a dynamic system governed by (1) can be derived, as shown in (62), as follows:

$$\omega_e = \frac{\pi\sqrt{\alpha + \beta A^2}}{2}\left(\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}t}{1 - m\sin^2 t}\right)^{-1}, \tag{62}$$

where

$$m = \frac{\beta A^2}{2(\alpha + \beta A^2)}. \tag{63}$$

The derivation of ω_e is presented in the appendix. It should be noted that ω_e contains an integral which can only be solved numerically in general. The maximum amplitude A of the oscillation satisfies $\beta A^2 = -\alpha$; the Duffing equation has a heteroclinic orbit with period $+\infty$ [40]. Hence, in order to avoid the heteroclinic orbit with period $+\infty$ for the soft spring properties in the Duffing equation in Examples 1 and 2, the nonlinear
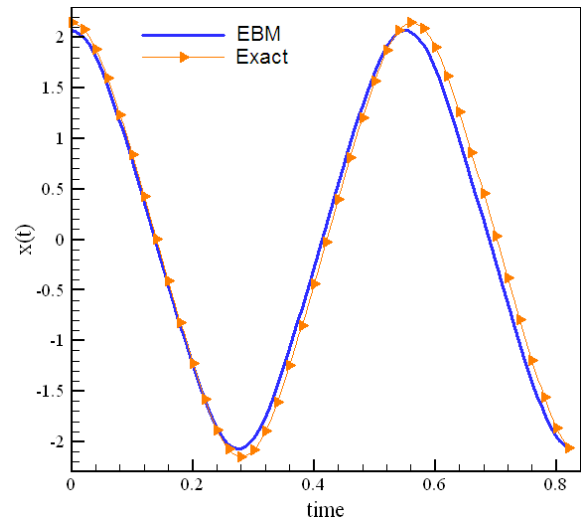

Fig. 3 (colour online). Comparison of the analytical approximates with the exact solution [3] for $k_1 = 5$, $k_2 = 5$, with $x(0) = 5$.

spring stiffness k_2 with the soft spring property in Example 1 should satisfy

$$k_2 > -\frac{k_1}{A^2}, \tag{64}$$

where $k_1 \in \mathbb{R}^+$ and $A \in \mathbb{R}$. From the Tables 1 and 2, the relative error of the energy balance method is 1.839% for the first-order analytical approximations, for $m = 1$, $k_1 = k_2 = 1$, $k_3 = 5$, $X_0 = 5$, and $Y_0 = 1$ in Example 1. In Example 2, the corresponding relative error is 2.102%, for $m = 1$, $k_1 = k_2 = 1$, $k_3 = 5$, $X_0 = 5$, and $Y_0 = 10$.
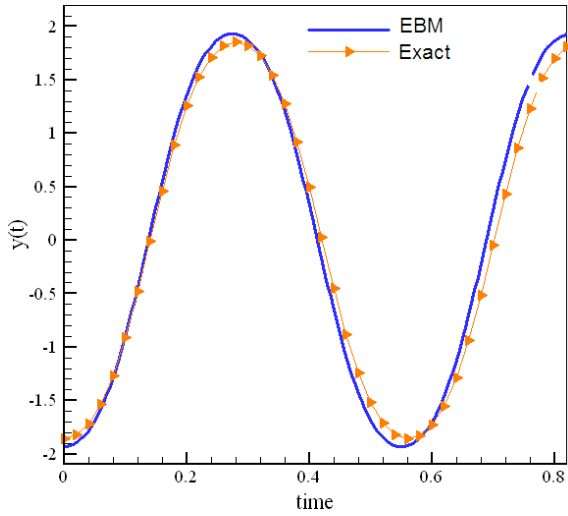
Fig. 4 (colour online). Comparison of the analytical approximates with the exact solution [3] for $k_1 = 5$, $k_2 = 5$, with $y(0) = 1$.
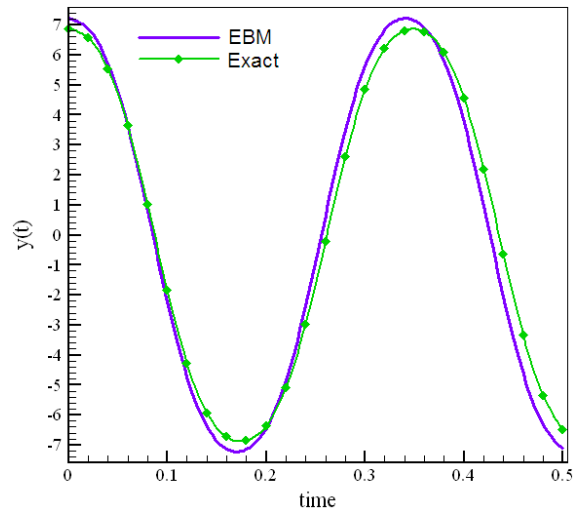

Fig. 6 (colour online). Comparison of the analytical approximates with the exact solution [3] for $k_1 = 5$, $k_2 = 5$, with $y(0) = -5$.
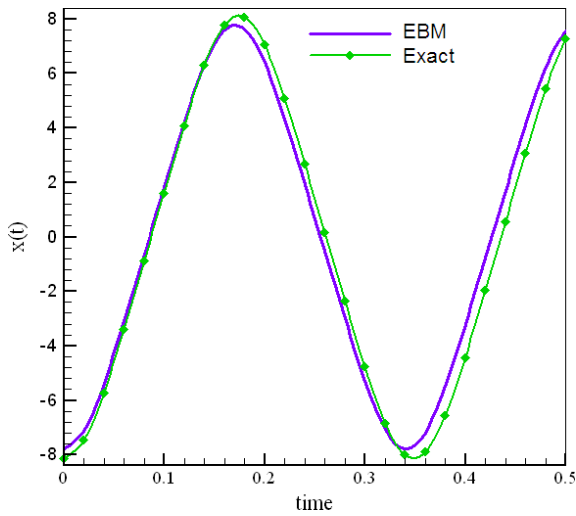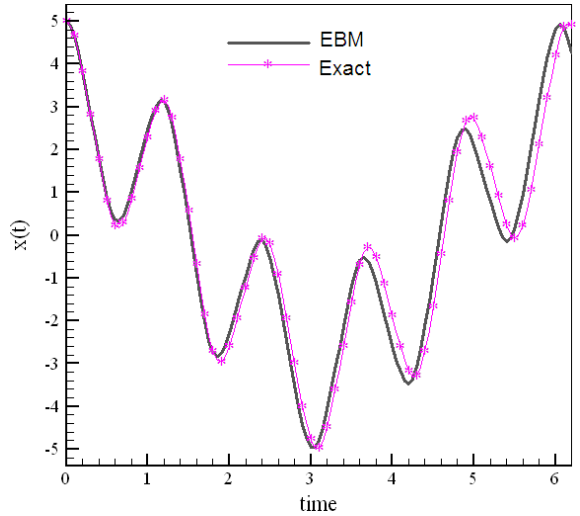

Fig. 5 (colour online). Comparison of the analytical approximates with the exact solution [3] for $k_1 = 5$, $k_2 = 5$, with $x(0) = 10$.

Fig. 7 (colour online) . Comparison of the analytical approximates with the exact solution [4] for $k_1 = 5$, $k_2 = 5$, $k_3 = 1$ with $x(0) = 5$.

The first-order approximate solution is of a high accuracy and the percentage error improves significantly from lower- to higher-order analytical approximations for different parameters and initial amplitudes. Hence, it is concluded that excellent agreement with the exact solutions for the nonlinear Duffing equation is provided.

To further illustrate and verify the accuracy of this approximate analytical approach, a comparison of the time history oscillatory displacement responses for the

two masses with exact solutions is depicted in Figures $3 - 10$. Figures $3 - 6$ represent the displacements $x(t)$ and $y(t)$ for the two masses with different initial conditions and spring stiffnesses in Example 1 while Figures $7 - 10$ represent the corresponding displacements $x(t)$ and $y(t)$ in Example 2. It is apparent that the first-order analytical approximations show excellent agreement with the exact solutions using the Jacobi elliptic functions. The displacement solutions for different analytical approximations for the two masses rely very
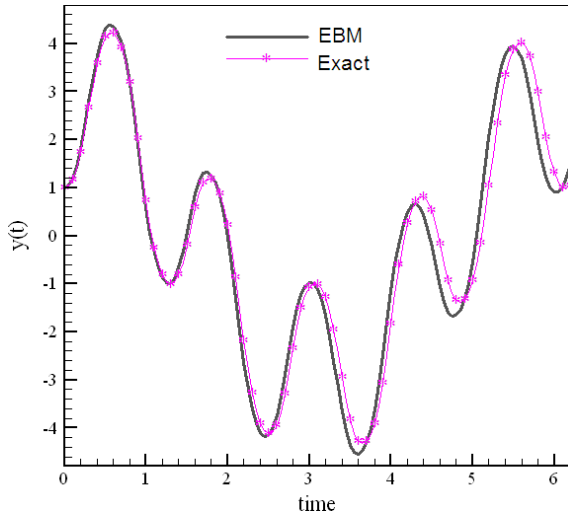
Fig. 8 (colour online). Comparison of the analytical approximates with the exact solution [4] for $k_1 = 5$, $k_2 = 5$, $k_3 = 1$ with $y(0) = 1$.
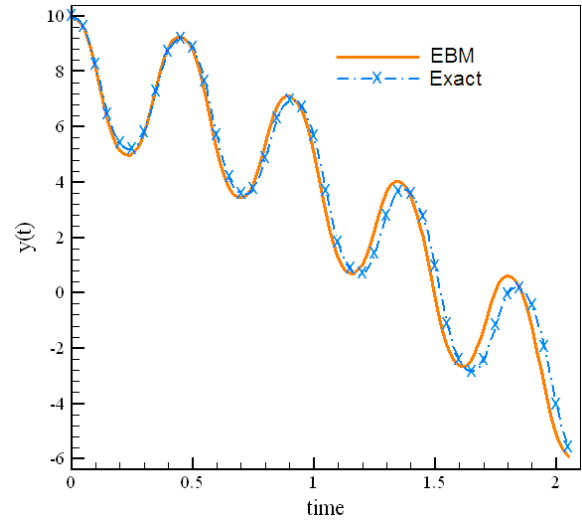

Fig. 10 (colour online). Comparison of the analytical approximates with the exact solution [4] for $k_1 = 1$, $k_2 = 1$, $k_3 = 5$ with $y(0) = 10$.
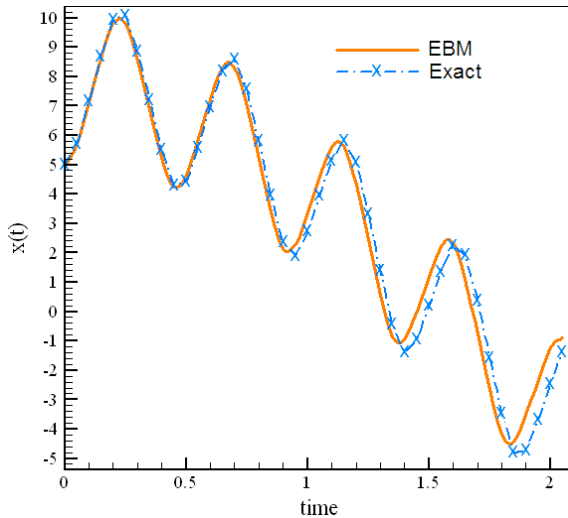

Fig. 9 (colour online). Comparison of the analytical approximates with the exact solution [4] for $k_1 = 5$, $k_2 = 5$, $k_3 = 5$ with $x(0) = 5$.

much on the masses m, and linear and nonlinear spring stiffnesses k_1, k_2, and k_3 in these examples.

5. Conclusion

The energy balance method (EBM) has been used to obtain the first-order approximate frequencies and periods for two degree of freedom oscillation systems. Excellent agreements between approximate frequen-

cies and the exact one have been demonstrated and discussed. In general, we conclude that this method is efficient for calculating periodic solutions for nonlinear oscillatory systems, and we think that the method has a great potential and could be applied to other strongly nonlinear oscillators.

Appendix

The exact solution of the dynamical system can be obtained by integrating the governing equation (1) and imposing the initial conditions (2) as follows. Equation (1) can be expressed as

$$\frac{1}{2}\dot{v}^2 + \frac{\alpha}{2}v^2 + \frac{\beta}{4}v^4 = C, \tag{A.1}$$

where C is a constant. Imposing the initial conditions in (2) yields

$$C = \frac{\alpha}{2}A^2 + \frac{\beta}{4}A^4. \tag{A.2}$$

Equating (A.1) and (A.2) yields

$$dt = \frac{dv}{\sqrt{\alpha(A^2 - v^2) + \frac{\beta}{2}(A^4 - v^4)}}. \tag{A.3}$$

Integrating (A.3), the period of oscillation is

$$T(A) = 4\int_0^A \frac{dv}{\sqrt{\alpha(A^2 - v^2) + \frac{\beta}{2}(A^4 - v^4)}}. \tag{A.4}$$

Substituting $v = -A\cos t$ into (A.4) and integrating gives

$$T(A) = \frac{4}{\sqrt{\alpha + \beta A^2}} \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}t}{\sqrt{1 - m\sin^2 t}}, \quad \text{(A.5)}$$

where

$$m = \frac{\beta A^2}{2(\alpha + \beta A^2)}. \quad \text{(A.6)}$$

The exact frequency ω_e is also a function of A and can be obtained from the period of the motion as

$$\omega_e(A) = \frac{\pi\sqrt{\alpha + \beta A^2}}{2} \left(\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}t}{1 - m\sin^2 t} \right)^{-1}. \quad \text{(A.7)}$$

[1] G. Chen, J. Sound Vib. **119**, 225 (1987).
[2] Ladygina and A. I. Manevich, J. Appl. Math. Mech. **57**, 257 (1993).
[3] L. Cveticanin, J. Sound Vib. **252**, 361 (2002).
[4] L. Cveticanin, J. Sound Vib. **247**, 279 (2001).
[5] A. F. Vakakis and R. H. Rand, Int. J. Nonlinear Mech. **39**, 1079 (2004).
[6] A. D. Dimarogonas and S. Haddad, Vibration for Engineers, Prentice-Hall, Englewood Cliffs, New Jersey 1992.
[7] S. Telli and O. Kopmaz, J. Sound Vib. **289**, 689 (2006).
[8] M. N. Hamdan and N. H. Shabaneh, J. Sound Vib. **199**, 711 (1997).
[9] S. Das, P. K. Gupta, V. S. Pandey, and A. D. K. N. Rai Z. Naturforsch. **65a**, 53 (2010).
[10] C. Chun, Z. Naturforsch. **65a**, 53 (2010).
[11] A. Barari, M. Omidvar, A. R. Ghotbi, and D. D. Ganji, Acta Appl. Math. **104**, 161 (2008).
[12] M. O. Miansari, M. E. Miansari, A. Barari, and G. Domairry, Int. J. Comput. Methods Eng. Sci. Mech. **11**, 79 (2010).
[13] J. H. He, Int. J. Nonlinear Mech. **37**, 309 (2002).
[14] H. Mirgolbabaei, A. Barari, L. B. Ibsen, and M. G. Sfahani, Archives Civil Mech. Eng. **10**, 41, 2010.
[15] S. T. Mohyud-Din, M. A. Noor, and K. I. Noor, Math. Probl. Eng. 2009 (2009), Article ID 234849, 25 pages, doi:10.1155/2009/234849.
[16] S. T. Mohyud-Din, M. A. Noor, and K. I. Noor, Int. J. Nonlinear Sci. Numer. Simul. **10**, 581 (2009).
[17] J. H. He, Commun. Nonlinear Sci. Numer. Simul. **4**, 81 (1999).
[18] N. Okuizumi and K. Kimura, J. Sound Vib. **272**, 675 (2004).
[19] M. Momeni, N. Jamshidi, A. Barari, and D. D. Ganji, Int. J. Comput. Math. DOI: 10.1080/00207160903337239.
[20] S. S. Ganji, D. D. Ganji, and Z. Z. Ganji, Acta Appl. Math. **106**, 79 (2009).
[21] J. H. He, Chaos, Solitons, and Fractals **34**, 1430 (2007).
[22] N. Tolou, S. H. A. Hashemi Kachapi, A. Barari, and D. D. Ganji, J. Appl. Functional Anal. **4**, 682 (2009).
[23] A. Konuralp, Comput. Math. Appl. **58**, 2152 (2009).
[24] F. Fouladi, E. Hosseinzadeh, A. Barari, and G. Domairry, J. Heat Transfer Res. **41**, 155 (2010).
[25] A. Barari, M. Omidvar, D. D. Ganji, and A. Tahmasebi Poor, J. Math. Probl. Eng. (2008), Article ID 394103, 1-13.
[26] S. T. Mohyud-Din, Z. Naturforsch. **65a**, 277 (2010).
[27] S. K. Lai, C. W. Lim, B. S. Wu, C. Wang, Q. C. Zeng, and X. F. He, Appl. Math. Model. **33**, 852 (2009).
[28] M. Omidvar, A. Barari, M. Momeni, and D. D. Ganji, Geomech. Geoeng. Int. J. **5**, 127 (2010).
[29] H. Babazadeh, G. Domairry, A. Barari, R. Azami, and A. G. Davodi, Frontiers Mech. Eng. China DOI: 10.1007/s11465-009-0033-x, 2010.
[30] L. B. Ibsen, A. Barari, and A. Kimiaeifar, Analysis of Highly Nonlinear Oscillation Systems Using He's Max-Min Method and Comparison with Homotopy Analysis and Energy Balance Methods, Sadhana, 2010, in press.
[31] M. G. Sfahani, S. S. Ganji, A. Barari, H. Mirgolbabaei, and G. Domairry, Analytical Solutions to Nonlinear Conservative Oscillator with Fifth-Order Nonlinearity, Earthquake Engineering and Engineering Vibration, 2010, in press.
[32] J. H. He, Int. J. Mod. Phys. B **20**, 1141 (2006).
[33] J. H. He, Int. J. Mod. Phys. B **22**, 3487 (2008).
[34] J. H. He, Mech. Res. Commun. **29**, 107 (2002).
[35] J. H. He, Phys. Rev. Lett. **90** (2003); Art. No. 174301.
[36] T. Ozis and A. Yildirim, Comput. Math. Appl. **54**, 1184 (2007).
[37] H. L. Zhang, Y. G. Xu, and J. R. Chang, Int. J. Nonlinear Sci. Numer. Simul. **10**, 207 (2009).
[38] N. Jamshidi and D. D. Ganji, Current Appl. Phys. **10**, 484 (2010).
[39] I. Mehdipour, D. D. Ganji, and M. Mozaffari, Current Appl. Phys. **10**, 104 (2010).
[40] E. A. Jackson, Perspectives of Nonlinear Dynamics, vol. 1, Cambridge University Press, UK, Cambridge 1989.